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Upper Bounds on the Critical Temperature for Kac Potentials

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We consider an Ising system in two dimensions with a two body ferromagnetic interaction $J_\gamma(x, y)$ that depends on the Kac scaling parameter γ. We prove that the inverse critical temperature $\beta_{cr}(\gamma)$ is strictly above the mean-field value (equal to 1), namely that there exists $C>0$ so that for any $b<C$, $\beta_{cr}(\gamma)>1+b\gamma^2\log\gamma^{-1}$ for all γ sufficiently small. The temperature shift $C\gamma^2\log\gamma^{-1}$ is to leading orders equal to the covariance of the magnetization fluctuations.

KEY WORDS: Kac potentials; critical temperature; fluctuations; Euclidean field theory.

1. INTRODUCTION

The phase diagram of the nearest neighbor Ising system is quite explicitly known, meanwhile we have only qualitative information on its structure when the interaction involves more sites. Paradoxically the problem is again "simple" if the range of the interaction is long. To give a quantitative meaning to this notion we follow Kac and introduce a fixed function $J(r)$, $r\in\mathbb{R}^d$, that we suppose smooth, spherically symmetric, supported by the ball of radius 1 and normalized to have total mass 1. We also suppose $J(r)\geqslant 0$, thus restricting to ferromagnetic interactions. We then define the spin-spin coupling strength between the spins at x and y, $x\neq y$ in $\mathbb{Z}^d$, as

$$J_\gamma(x, y)=c_\gamma\gamma^d J(\gamma(y-x)) \tag{1.1}$$

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where c_γ is a normalization coefficient that goes to 1 as $\gamma \to 0^+$:

$$c_\gamma := \left(\gamma^d \sum_{y \neq 0} J(\gamma y) \right)^{-1} \tag{1.2}$$

Thus the formal hamiltonian of a spin configuration $\sigma = \{\sigma(x), x \in \mathbb{Z}^d\}$, $\sigma(x) = \pm 1$, is

$$H_\gamma(\sigma) = -\tfrac{1}{2} \sum_{x \neq y} J_\gamma(x, y)\, \sigma(x)\, \sigma(y) \tag{1.3}$$

We can now give a precise mathematical meaning to a statement about a property holding if the interaction range is long, meaning that there is $\gamma^* > 0$ so that for all $\gamma \leqslant \gamma^*$ that property holds for the hamiltonian (1.3). The question we are interested in is the presence or absence of phase transitions when γ is small but positive, i.e., long but finite range interactions. We know since the works of Kac, Uhlenbeck and Hemmer,[7] and specifically of Lebowitz and Penrose,[8] that the phase diagram of (1.3) in the limit $\gamma \to 0^+$ converges to the mean field phase diagram which has a very simple structure: two phases (the plus and the minus phase) when $\beta > 1$ and one phase (with 0 magnetization) when $\beta \leqslant 1$. The mean field inverse critical temperature β^{mf} is thus equal to 1.

This beautiful convergence result unfortunately does not answer our question. In fact the diagram of the free energies (versus the magnetization, after the thermodynamical limit, at $\gamma > 0$ and fixed temperature) might very well converge as $\gamma \to 0^+$ to the mean field diagram, which at $\beta > 1$ has a flat part corresponding to phase coexistence, without having, before the limit, any flat part or maybe flat portions elsewhere than in the limit. The mere convergence of the free energies does not allow to conclude that if there is phase transition after the limit then there is phase transition when γ is non zero, a statement clearly false in $d = 1$ dimensions.

The region $\{\beta < 1\}$ however has a unique phase also when $\gamma > 0$. Miracolously in fact the Dobrushin uniqueness condition

$$\beta \sum_{y \neq x} J_\gamma(x, y) < 1 \tag{1.4}$$

coincides with the mean field condition $\beta < \beta^{\mathrm{mf}}$, even though (1.4) involves only the energy and not the other thermodynamic potentials (like the entropy) that enter in the determination of the critical temperature. Indeed this coincidence (between the Dobrushin condition and the mean field critical temperature) fails if we extend our considerations to systems where the spins have more general values, hence with different entropy functions.

The region $\{\beta > 1\}$ is studied in ref. 4 where it is proved that in $d \geqslant 2$ for any $\beta > 1$ there is $\gamma_\beta > 0$ such that for all $\gamma \leqslant \gamma_\beta$ there are at least two Gibbs states. From the above considerations we can then conclude that the true inverse critical temperature $\beta_{cr}(\gamma)$ is not smaller than 1 and converges to 1 as $\gamma \to 0^+$.

It thus only remains to investigate the phase diagram in an infinitesimal (as $\gamma \to 0^+$) neighborhood of $\beta = 1$ and to locate more accurately the position of the inverse critical temperature inside this neighborhood. This has been partially done in ref. 2 for an Ising system in $d = 3$, showing that $\beta_{cr}(\gamma) \leqslant 1 + 0(\gamma^2)$ (where, following the original proposal of Kac, the interaction is chosen in such a way that it may be transformed into a nearest neighbor system for which reflection positivity applies). In this paper we consider $d = 2$, a general interaction as in (1.3) and work on the other side of the inequality proving, as announced in ref. 3, that the true inverse critical temperature is strictly above 1:

Theorem 1.1. For any $b < C$,

$$C := \frac{1}{\pi D}, \qquad D := \int_{\mathbb{R}^2} dr\, J(r)\, r^2 \tag{1.5}$$

there is $\gamma(b) > 0$ so that for all $0 < \gamma \leqslant \gamma(b)$, $\beta_{cr}(\gamma) \geqslant 1 + b\gamma^2 \log \gamma^{-1}$.

We prove Theorem 1.1 via a perturbative analysis of the hierarchy of equations for the correlation functions which gives a very detailed understanding of the structure of the system up to $\beta \approx 1$. In particular we have a simple interpretation of the temperature shift found in Theorem 1.1 in terms of the magnetization density fluctuations. The term $C\gamma^2 \log \gamma^{-1}$ is in fact to leading orders the covariance C_γ of the magnetization fluctuations. Their magnitude goes like γ^2 when $\beta < 1$, the extra divergent factor $\log \gamma^{-1}$ appears when β approaches 1 and plays a leading role in the temperature shift, as explained in the following heuristic argument. The original mean field equation is

$$m = \tanh \beta m \tag{1.6}$$

which expanded around $m = 0$ (β is fixed in a small neighborhood of 1) gives

$$m = \beta m - \frac{\beta^3}{3} m^3 + R_m \tag{1.7}$$

R_m the remainder term. In (1.7) m^3 is an approximation for the expectation of the cube of the empirical magnetization S of a suitable block of spins.

Writing $S = m + Y$, with Y the fluctuation field around the average magnetization m, we have $\langle S^3 \rangle = m^3 + 3m\langle Y^2 \rangle$, supposing that the other terms (odd in Y) vanish by symmetry (or that they can be neglected). As we are interested in the transition through $m = 0$ we may certainly suppose that $m^2 \ll \langle Y^2 \rangle$, hence the leading contribution to $\langle S^3 \rangle$ is $3mC_\gamma$, $C_\gamma = \langle Y^2 \rangle$ (the anomalously large fluctuations, i.e., the presence of the factor $\log \gamma^{-1}$, is used to make the argument rigorous). Taking this into account we arrive to a new mean field equation:

$$m = \beta m - \beta^3 C_\gamma m + R'_m \tag{1.8}$$

with R'_m a new remainder term. By ferromagnetic inequalities the covariance C_γ is an increasing function of β and its value at $\beta = 1 - \gamma^2$ (i.e. still in the one phase region) can be "easily" estimated as

$$C_\gamma \approx C\gamma^2 \log \gamma^{-1} \tag{1.9}$$

which together with (1.8) yields a lower bound on the inverse critical temperature which is just the same as that proved in Theorem 1.1. The logarithmic divergence in (1.9) is typical of $d = 2$, in $d = 1$ there is no divergence and the whole argument fails.

We actually find in the course of the proofs that the covariance (of the magnetization fluctuations) does not increase past its value at $\beta = 1 - \gamma^2$ (to leading orders in γ) till $\beta < 1 + C\gamma^2 \log \gamma^{-1}$. Supposing that this remains true also for slightly larger values of β, according to the previous heuristic arguments we would then conclude that the magnetization should become positive when $\beta = 1 + C'\gamma^2 \log \gamma^{-1}$ with any $C' > C$ and γ correspondingly small. This would show that $1 + C\gamma^2 \log \gamma^{-1}$ is the inverse critical temperature on the scale $\gamma^2 \log \gamma^{-1}$, as $\gamma \to 0^+$. Unfortunately our approach does not give indications on the validity of this conjecture which therefore remains a totally open problem.

While in this paper we have focused our attention on the shift of the critical temperature caused by the magnetic fluctuations, in a forthcoming paper we reverse this viewpoint centering our analysis on the fluctuation field itself proving its convergence to a ϕ^4 Euclidean field theory as $\gamma \to 0^+$ and with $\beta \approx 1 + C\gamma^2 \log \gamma^{-1}$. The origin of the Wick regularization in the limit theory is then explained in terms of the critical temperature shift found in Theorem 1.1, which is in fact recognized as the microscopic origin of the Wick regularization.

Our results are obtained starting from the DLR equations and deriving a system of coupled equations for the correlation functions that expresses a given n-points correlation function in terms of all the others. By a truncation procedure we get a hierarchical structure that allows to describe a given n-points correlation function in terms of k-points correlations with $k \leqslant n+2$ plus a negligible error. By suitably rearranging the terms (i.e. by writing them via the truncated correlation functions) we get equations that can be solved iteratively with an error that for $\beta < 1$ is controllable and the iteration procedure converges. The hard part is to extend the analysis beyond $\beta = 1$. We achieve this by making an ansatz on bounds on the correlation functions and then proving that the ansatz is consistent with the above mentioned equations. To conclude we need to show that the actual correlation functions satisfy the ansatz and this is done by a continuity argument starting from $\beta < 1$. In this part we use the Newman's Gaussian inequalities and the convexity properties of the pressure to prove uniqueness of the even correlation functions to extend the bounds proved for the Gibbs measure with periodic conditions to the Gibbs measure with plus boundary conditions.

In Section 2 we apply these considerations to the even correlation functions and in Section 3 to the odd ones, in particular to the magnetization itself, thus proving Theorem 1.1. In an Appendix we study the central limit theorem for the iterates of the transition probability $J_\gamma(x, y)$, deriving estimates that are uniform in γ.

2. BOUNDS ON THE EVEN CORRELATION FUNCTIONS

We consider an auxiliary spin-spin interaction. Let $\mathscr{L}$ be the collection of all $\lambda = \{\lambda_{0,x}, x \in \mathbb{Z}^2 \backslash 0\}$ in $[0, 1]^{\mathbb{Z}^2 \backslash 0}$. For $\lambda \in \mathscr{L}$ and $y \neq x$ in $\mathbb{Z}^2$, we set $\lambda_{x,y} = \lambda_{0,y-x}$, and

$$I_\gamma(x, y) := J_\gamma(x, y) + \gamma^{100} \lambda_{x,y} e^{-|x-y|} \tag{2.1}$$

In the sequel Λ will always denote a torus in $\mathbb{Z}^2$ (centered at 0) and for $y \neq x$ in Λ we write

$$I_{\gamma,\Lambda}(x, y) := \sum_{z \sim y} I_\gamma(x, z) \tag{2.2}$$

where the sum is over all $z \in \mathbb{Z}^2$ such that $z = y$ modulo Λ.

The expectation of the Gibbs measure with interaction (2.1)–(2.2) in the torus Λ (i.e. in the cube Λ with periodic boundary conditions) at the inverse temperature β is denoted by $\langle \cdot \rangle_{\beta,\gamma,\lambda,\Lambda}$ or simply $\langle \cdot \rangle$ when the values of the parameters are clear from the context.

We shorthand for $\gamma > 0$ and $r \in \mathbb{R}^2$

$$\delta_\gamma(r) := \frac{1}{2} \max\left\{1, \log \frac{\gamma^{-2}}{1 + (\gamma r)^2}\right\} \tag{2.3}$$

namely $\delta_\gamma(r) \approx \log \gamma^{-1}$ for r small whereas it reaches for $r \approx \gamma^{-2}$ a plateau with value 1/2. For $b \in \mathbb{R}$ we set

$$\beta_{b,\gamma} := 1 + b\gamma^2 \log \gamma^{-1} \tag{2.4}$$

and consider hereafter only $b < C$ (C as in Theorem 1.1).

Definition 2.1. β is (K, γ)-good, $\gamma > 0$, $K > 0$, relative to $\lambda \in \mathscr{L}$ and to a torus Λ if for any $x \neq 0$ in Λ

$$\langle \sigma(0)\, \sigma(x) \rangle_{\beta,\gamma,\lambda,\Lambda} \leqslant K\gamma^2\, \delta_\gamma(x) \tag{2.5}$$

β is (K, γ)-good if it is (K, γ)-good for any λ and any torus Λ whose side is not smaller than γ^{-2}.

By ferromagnetic inequalities if β is (K, γ)-good relative to λ and Λ, any $\beta' \leqslant \beta$ is also (K, γ)-good relative to λ and Λ. The main technical result of this Section is:

Proposition 2.2. For any $b < C$ and $\varepsilon > 0$ there is $\gamma_{b,\varepsilon} > 0$ so that if $\gamma \leqslant \gamma_{b,\varepsilon}$ and $\beta_{b,\gamma}$ is $(100C, \gamma)$-good relative to λ and to a torus Λ of side not smaller than γ^{-2}, then $\beta_{b,\gamma}$ is $((1+\varepsilon)\, C, \gamma)$-good relative to λ and to Λ.

We postpone both the proof of Proposition 2.2 and the simple proof (see Lemma 2.6 below) that there is $\gamma_0 > 0$ so that $\beta = 1/2$ is (C, γ)-good for any $\gamma \leqslant \gamma_0$. Then, by the continuity in β of the functions $\langle \sigma(0)\, \sigma(x) \rangle_{\beta,\gamma,\lambda,\Lambda}$ we obtain:

Corollary 2.3. Let $b < C$, $\varepsilon > 0$ and $\gamma'_{b,\varepsilon} := \min(\gamma_{b,\varepsilon}, \gamma_0)$. Then for any $\gamma \leqslant \gamma'_{b,\varepsilon} \beta_{b,\gamma}$ is $((1+\varepsilon)\, C, \gamma)$-good.

Denote by $\langle \cdot \rangle_{\beta,\gamma,+}$ the expectation with respect to the + Gibbs measure on the whole $\mathbb{Z}^2$ with the pure interaction J_γ, i.e. with $\lambda \equiv 0$. We will next prove:

Theorem 2.4. Let b, ε and $\gamma'_{b,\varepsilon}$ be as in Corollary 2.3. Then for any $\gamma \leqslant \gamma'_{b,\varepsilon}$ and any $x \neq 0$

$$\langle \sigma(0)\, \sigma(x) \rangle_{\beta_{b,\gamma},\gamma,+} \leqslant (1+\varepsilon)\, C\gamma^2\, \delta_\gamma(x) \tag{2.6}$$

Proof. Let $x \neq 0$, fix $\{\lambda^*_{0,y}, y \in \mathbb{Z}^2 \backslash \{0, x\}\}$ in $[0, 1]^{\mathbb{Z}^2 \backslash \{0, x\}}$ and let $\lambda_{0,x}$ vary in $[0, 1]$. Then the limit thermodynamic pressure P is a function of the only variable $\lambda_{0,x}$, $P = P(\lambda_{0,x})$. By convexity there is $\lambda^*_{0,x} \in (0, 1)$ where $P(\lambda_{0,x})$ is differentiable. We call $\lambda^* \in \mathscr{L}$ the interaction completed with $\lambda_{0,x} = \lambda^*_{0,x}$. By Theorem 7.3.2 in Ruelle,[10] if μ and ν are any two translationally invariant Gibbs measures with interaction determined by λ^*, then

$$\mathbb{E}_\mu(\sigma(0)\, \sigma(x)) = \mathbb{E}_\nu(\sigma(0)\, \sigma(x))$$

In particular this holds with ν the + Gibbs measure and μ any weak limit point of Gibbs measures on Λ (with periodic boundary conditions). By ferromagnetic inequalities the correlation functions in the + state do not increase if we drop the interaction λ^*, thus

$$\begin{aligned}\langle \sigma(0)\, \sigma(x)\rangle_{\beta_h, \gamma, \gamma, +} &\leq \mathbb{E}_\nu(\sigma(0)\, \sigma(x)) = \mathbb{E}_\mu(\sigma(0)\, \sigma(x))\\ &\leq (1+\varepsilon)\, C\gamma^2\, \delta_\gamma(x)\end{aligned}$$

because μ is the limit of measures that by assumption satisfy the last inequality. Theorem 2.4 is proved. ∎

The remaining of this Section is devoted to the proof of Proposition 2.2. We begin with a well known mean field bound on the two body correlation functions whose proof is reported for the sake of completeness.

Lemma 2.5. For any β, γ, λ and Λ and any $x \neq 0$ in Λ

$$\langle \sigma(0)\, \sigma(x)\rangle_{\beta, \gamma, \lambda, \Lambda} \leq \beta I_{\gamma, \Lambda}(x, 0) + \sum_{y \in \Lambda \backslash 0} I_{\gamma, \Lambda}(x, y) \langle \sigma(0)\, \sigma(y)\rangle_{\beta, \gamma, \lambda, \Lambda} \tag{2.7}$$

Proof. Dropping the subfixes $\beta, \gamma, \lambda, \Lambda$ from the expectation and denoting by $\mathbb{E}_{\sigma(0) = a}$, $a = \pm 1$, the conditional expectation given $\sigma(0) = a$, by symmetry

$$\langle \sigma(0)\, \sigma(x)\rangle = \mathbb{E}_{\sigma(0) = 1}(\sigma(x))$$

Let $\mathbb{E}_{\sigma_{\Lambda \backslash x}}(\cdot)$ be the conditional expectation given the value of the spin configuration $\sigma_{\Lambda \backslash x}$ in $\Lambda \backslash x$. We have

$$\langle \sigma(0)\, \sigma(x)\rangle = \mathbb{E}_{\sigma(0) = 1}(\mathbb{E}_{\sigma_{\Lambda \backslash x}}(\sigma(x))) - \mathbb{E}(\mathbb{E}_{\sigma_{\Lambda \backslash x}}(\sigma(x)))$$

(the last term is 0 by symmetry). Let $\mathcal{Q} = \mathcal{Q}(\sigma_{\Lambda \backslash x}, \sigma'_{\Lambda \backslash x})$ be a probability on $\{-1, 1\}^{\Lambda \backslash x} \times \{-1, 1\}^{\Lambda \backslash x}$ such that its first marginal is the Gibbs measure on $\Lambda \backslash x$ conditioned on $\sigma(0) = 1$ while the second marginal is the unconditioned Gibbs measure on $\Lambda \backslash x$. $\mathcal{Q}$ is then called a joint representation of these two measures. By an abuse of notation we also denote by $\mathcal{Q}(\cdot)$ the expectation with respect to $\mathcal{Q}$, we then have

$$\langle \sigma(0)\, \sigma(x)\rangle = \mathcal{Q}(\mathbb{E}_{\sigma_{\Lambda \backslash x}}(\sigma(x) - \mathbb{E}_{\sigma'_{\Lambda \backslash x}}(\sigma(x))))$$

By FKG it is possible to choose $\mathcal{Q}$ so that

$$\mathcal{Q}(\sigma_{\Lambda \backslash x} \geq \sigma'_{\Lambda \backslash x}) = 1 \tag{2.8}$$

We then use the DLR equations to write

$$|\mathbb{E}_{\sigma_{\Lambda\setminus x}}(\sigma(x)) - \mathbb{E}_{\sigma'_{\Lambda\setminus x}}(\sigma(x))|$$
$$= |\tanh(\beta h_\gamma(x)) - \tanh(\beta h'_\gamma(x))| \leqslant \beta\, |h_\gamma(x) - h'_\gamma(x)|$$

where

$$h_\gamma(x) := \sum_{y \in \Lambda\setminus x} I_{\gamma,\Lambda}(x, y)\, \sigma(y) \tag{2.9}$$

and $h'_\gamma(x)$ is defined with $\sigma'(y)$ instead of $\sigma(y)$. By (2.8)

$$\langle \sigma(0)\, \sigma(x)\rangle \leqslant \mathscr{Q}\left(\beta \sum_{y \in \Lambda\setminus x} I_{\gamma,\Lambda}(x, y)[\sigma(y) - \sigma'(y)]\right)$$
$$= \beta \sum_{y \in \Lambda\setminus x} I_{\gamma,\Lambda}(x, y)(\mathbb{E}_{\sigma(0)=1}(\sigma(y)) - \mathbb{E}(\sigma(y)))$$

(because $\mathscr{Q}$ is a joint representation). By symmetry the last term drops and we obtain (2.7). Lemma 2.5 is proved. ∎

Lemma 2.6. For any $\varepsilon > 0$ there is $\gamma_\varepsilon > 0$ so that $\beta = 1/2$ is (ε, γ)-good for any $\gamma \leqslant \gamma_\varepsilon$.

Proof. Let $\langle \cdot \rangle \equiv \langle \cdot \rangle_{\beta,\gamma,\lambda,\Lambda}$ with $\beta = 1/2$. By (2.7) and recalling (2.1)

$$\langle \sigma(0)\, \sigma(x)\rangle \leqslant 2^{-1} J_{\gamma,\Lambda}(x, 0) + 2^{-1} \sum_{y \notin \Lambda\setminus 0} J_{\gamma,\Lambda}(x, y) \langle \sigma(0)\, \sigma(y)\rangle + c\gamma^{100}$$

with c a suitable constant. By iteration

$$\langle \sigma(0)\, \sigma(x)\rangle \leqslant \sum_{n \geqslant 1} 2^{-n} J^n_{\gamma,\Lambda}(x, 0) + 2c\gamma^{100}$$

Since $J_\gamma(0, x) = 0$ if $\gamma\, |x| \geqslant 1$

$$\langle \sigma(0)\, \sigma(x)\rangle \leqslant \sum_{n \geqslant \gamma\, |x|} 2^{-n} J^n_{\gamma,\Lambda}(x, 0) + 2c\gamma^{100}$$

Moreover recalling that for any $n \geqslant 1$ the sum over x of $J^n_\gamma(0, x)$ (J^n_γ the nth convolution of J_γ) is equal to 1 and that the side of Λ is $\geqslant \gamma^{-2}$, we have

$$J^n_{\gamma,\Lambda}(x, 0) = \sum_{y \in \Lambda} J^{n-1}_{\gamma,\Lambda}(x, y)\, J_{\gamma,\Lambda}(y, 0) \leqslant \sup_{y \neq 0} J_\gamma(y, 0) \leqslant c'\gamma^2 \tag{2.10}$$

with c' a suitable constant. Thus

$$\langle \sigma(0)\,\sigma(x)\rangle \leqslant 2^{-\gamma |x|} c'' \gamma^2 + 2c\gamma^{100}$$

with c'' a suitable constant. Lemma 2.6 is proved. ∎

The bound (2.7) will be used when $\beta < 1$, for $\beta \geqslant 1$ we will need a more refined analysis. From bounds on the two point correlation functions we gain bounds on correlations with any even number of points via the Newman's Gaussian inequalities, that we recall after some new notation.

Notation 2.7. $\Lambda^k, k \geqslant 1$, denotes the family of all the subsets Y in Λ with k elements.

$$\sigma(Y) := \prod_{y \in Y} \sigma(y),\ I_{\gamma,\Lambda}(x, Y) := \prod_{y \in Y} I_{\gamma,\Lambda}(x, y) \tag{2.11}$$

$J_{\gamma,\Lambda}(x, Y)$ is defined analogously. We will write S^k when $\Lambda \equiv \mathbb{Z}^2$.

Theorem 2.8 (Newman's Gaussian inequalities). For any β, γ, λ, Λ, any $n \geqslant 1$ and any $Y \subset \Lambda^{2n}$, setting $\langle \cdot \rangle \equiv \langle \cdot \rangle_{\beta,\gamma,\lambda,\Lambda}$,

$$\langle \sigma(Y)\rangle \leqslant \sum_{\{Y_1,\ldots,Y_n\}} \langle \sigma(Y_1)\rangle \cdots \langle \sigma(Y_n)\rangle \tag{2.12}$$

where the sum is over all the partitions of the set Y into atoms $Y_1,\ldots, Y_n$ of two elements each.

Theorem 2.8 is a particular case of the Newman's Gaussian inequalities for general ferromagnetic interactions,[9] which include the case of the odd correlations in the infinite volume plus Gibbs state, see Theorem 3.3 below.

As remarked earlier Lemma 2.6 becomes useless when β increases past 1. The analysis of the case $\beta > 1$ is based on a very accurate study of the hierarchy of the correlation functions and it is at the heart of the proof of Proposition 2.2. To simplify notation we fix β, γ, λ and Λ and write $\langle \cdot \rangle$ for the expectation of the Gibbs measures with such parameters. Recalling (2.9) we use the DLR equations to write for any $x \neq 0$ in Λ

$$\langle \sigma(0)\,\sigma(x)\rangle = \langle \sigma(0) \tanh(\beta h_\gamma(x))\rangle \tag{2.13}$$

By a Taylor expansion of the hyperbolic tangent the right hand side can be expressed in terms of a series of correlation functions. Using the assumption that β is (K, γ)-good and the Gaussian inequalities we will truncate the expansion controlling the error in terms of two point correlations. It turns out that if the Taylor expansion is truncated after the four point correlations, the contribution of the remainder can be neglected. This is shown in the next Lemma where, after the truncation, we retain only the two and four point correlations. Because of these latter the resulting equation is not a closed equation for the two point correlations but the terms can be rearranged in such a way that the four point correlations appear as truncated correlation functions, a feature that will be crucial for the successive analysis. We will comment further this point after the lemma. We need first a few new notation:

$$q_\gamma := \beta^3 \sum_{Y \in S^2} J_\gamma(0, Y) \langle \sigma(Y) \rangle \tag{2.14}$$

(which depends also on β and λ)

$$v_\gamma := q_\gamma + \beta^3 \sum_{x \in \mathbb{Z}^2} J_\gamma(0, x)^2 \tag{2.15}$$

and, for $Y \in \Lambda^4$,

$$\langle \sigma(Y) \rangle^T := \langle \sigma(Y) \rangle - \sum_{\{Y_1, Y_2\}} \langle \sigma(Y_1) \rangle \langle \sigma(Y_2) \rangle \tag{2.16}$$

the sum being over the three partitions of Y into two atoms of two elements each. (2.16) is the four point truncated correlation function. We use below the shorthand notation

$$a = b \pm c \Leftrightarrow |a - b| \leqslant c \tag{2.17}$$

Lemma 2.9. There is $c > 0$ so that for any $\gamma > 0$, any torus Λ of side $\geqslant \gamma^{-2}$, any $\lambda \in \mathscr{L}$, any $\beta \leqslant 2$ which is $(100C, \gamma)$-good relative to λ and Λ and any $x \in \Lambda \backslash 0$

$$\begin{aligned} \langle \sigma(0)\, \sigma(x) \rangle = {} & (\beta - q_\gamma)\, J_\gamma(x, 0) + (\beta - v_\gamma) \sum_{y \neq 0} J_\gamma(x, y) \langle \sigma(0)\, \sigma(y) \rangle \\ & - \beta^3 \sum_{\substack{0 \notin Y \\ Y \in S^3}} J_\gamma(x, Y) \langle \sigma(0)\, \sigma(Y) \rangle^T \\ & \pm c(\gamma^4 + [\gamma \log \gamma^{-1}]^5) \end{aligned} \tag{2.18}$$

Proof. A Taylor–Lagrange expansion to fifth order of the hyperbolic tangent in (2.13) gives

$$\begin{aligned}\langle\sigma(0)\,\sigma(x)\rangle = {} & \beta J_\gamma(x,0)+\beta\sum_{y\neq 0}J_\gamma(x,y)\langle\sigma(0)\,\sigma(y)\rangle\\ & -\frac{\beta^3}{3}\sum_{Y\in(\Lambda\backslash 0)^3}J_\gamma(x,Y)\langle\sigma(0)\,\sigma(Y)\rangle\\ & -\beta^3\sum_{y\neq 0}J_\gamma(x,y)\langle\sigma(0)\,\sigma(y)\rangle\left(\sum_{z\neq y}J_\gamma(x,z)^2\right)\\ & -\beta^3 J_\gamma(x,0)\sum_{Y\in(\Lambda\backslash 0)^2}J_\gamma(x,Y)\langle\sigma(Y)\rangle\\ & -\frac{\beta^3}{3}\sum_{y\neq 0}J_\gamma(x,y)^3\langle\sigma(0)\,\sigma(y)\rangle-\beta^3 J_\gamma(x,0)\sum_{y\neq 0}J_\gamma(x,y)^2\\ & -\frac{\beta^3}{3}J_\gamma(x,0)^3\pm c(\langle h_\gamma(x)^{10}\rangle^{1/2}+\gamma^{100}) \qquad (2.19)\end{aligned}$$

The last bracket term takes into account the remainder term of the Taylor–Lagrange expansion and the error due to changing $I_{\gamma,\Lambda}$ into J_γ in the other terms. The latter is bounded by $c\gamma^{100}$ if c is a suitable constant. The former is obtained by using Cauchy–Schwartz and bounding by a constant the sup norm of the fifth derivative of the hyperbolic tangent. The first row on the right hand side of (2.19) refers to the first order terms of the expansion distinguishing coincident and different points. The second order terms of the expansion are absent, the third order terms are written in the second to fifth row of (2.19), distinguishing the various cases of different and equal sites.

The first term after the equality and the first term in the fourth row reconstruct $(\beta-q_\gamma)\,J_\gamma(x,0)$ with an error as in (2.18), because

$$J_\gamma(x,0)\left|\beta^3\sum_{Y\in(\Lambda\backslash 0)^2}J_\gamma(x,Y)\langle\sigma(Y)\rangle-q_\gamma\right|\leqslant c\gamma^2[\gamma^2 100C\gamma^2\log\gamma^{-1}] \qquad (2.20)$$

In fact by (2.10) $J_\gamma(x,0)\leqslant c'\gamma^2$, c' a suitable constant, and the two point correlations are bounded using the assumption that β is $(100C,\gamma)$-good.

By (2.16) the term in the second row can be written as

$$\begin{aligned}&-\frac{\beta^3}{3}\sum_{Y\in(\Lambda\backslash 0)^3}J_\gamma(x,Y)\langle\sigma(0)\,\sigma(Y)\rangle\\ &\quad=-\frac{\beta^3}{3}\sum_{Y\in(\Lambda\backslash 0)^3}J_\gamma(x,Y)\langle\sigma(0)\,\sigma(Y)\rangle^T\\ &\qquad-\beta^3\sum_{y\neq 0}J_\gamma(x,y)\langle\sigma(0)\,\sigma(y)\rangle\left[\sum_{Y\in\Lambda^2}J_\gamma(x,Y)\langle\sigma(Y)\rangle\right]\\ &\qquad\pm c\gamma^2[100C\gamma^2\log\gamma^{-1}]^2 \qquad (2.21)\end{aligned}$$

The error term takes into account the cases when $Y \cap \{0, y\} \neq \varnothing$ and it is bounded analogously to that in (2.20). With (2.21) we have thus recovered also the truncated correlation function of (2.18). The second term in the first row of (2.19) and the term in the third row of (2.19) together with the first term in the second row of (2.21) reconstruct the term with the two point correlations in (2.18) with an error compatible with (2.18). This comes from the constraint $z \neq y$ in the sum in (2.19). The error when dropping such a constraint is bounded by γ^4 times $100C\gamma^2 \log \gamma^{-1}$ which comes from the two point correlation. The error is thus compatible with (2.18).

The second term in the fourth row and the first two in the fifth row are respectively bounded by

$$c\gamma^4[100C\gamma^2 \log \gamma^{-1}],\ c\gamma^4,\ c\gamma^6$$

with c a suitable constant (having used again (2.10) and that β is $(100C, \gamma)$-good).

Finally we expand the term with $h_\gamma(x)^{10}$ into a sum of products of spins getting

$$\langle h_\gamma(x)^{10} \rangle \leqslant c \sum_{n=1}^{5} \gamma^{10-2n} \sup_{Y \in \Lambda^{2n}} \langle \sigma(Y) \rangle$$

with c a suitable constant. The bound comes from distinguishing the number of points in the resulting correlation functions and recalling that J_γ is bounded proportionally to γ^2 and that the sum of $J_\gamma(0, x)$ is equal to 1. By the Gaussian inequalities and the assumption that β is $(100C, \gamma)$-good we then prove that the last term in (2.19) is bounded proportionally to $[\gamma^2 \log \gamma^{-1}]^5$. Lemma 2.9 is proved. ∎

Two are the crucial features in (2.18): the factor $\beta - v_\gamma$ in front of the two point correlation functions and that the four point correlations appear as truncated correlation functions. We will prove in Proposition 2.10 that v_γ is bounded from below proportionally to $\gamma^2 \log \gamma^{-1}$ and, Proposition 2.11, that the four point truncated correlation functions are bounded from above proportionally to $\gamma^4 \log \log \gamma^{-1}$. Observe that this bound is much smaller than the a priori bound obtained from the two point correlations using Gaussian inequalities, which in fact gives $[\gamma^2 \log \gamma^{-1}]^2$. This will allow to neglect the term with the truncated correlation function and (2.18) will then become a closed equation for the two point correlations that can and will indeed be solved. The inverse temperature is replaced in this equation by $\beta - v_\gamma$ which amounts to an effective reduction of the inverse temperature by v_γ yielding in the end Proposition 2.2.

Proposition 2.10. For any $\varepsilon > 0$ there is $\gamma_\varepsilon > 0$ so that for any $\lambda \in \mathscr{L}$, any $\gamma \leqslant \gamma_\varepsilon$, any torus Λ of side $\geqslant \gamma^{-2}$, any $\beta \geqslant 1 - \gamma^2(\log \gamma^{-1})^{100}$ and any $0 < |x| \leqslant 100\gamma^{-1}$ (recall that C is defined in (1.5))

$$\langle \sigma(0)\, \sigma(x) \rangle_{\beta, \gamma, \lambda, \Lambda} \geqslant (C - \varepsilon)\, \gamma^2 \log \gamma^{-1} \tag{2.22}$$

Proof. By ferromagnetic inequalities we may and will restrict to $\beta = \beta_\gamma := 1 - \gamma^2(\log \gamma^{-1})^{100}$ and in the course of the proof $\langle \cdot \rangle \equiv \langle \cdot \rangle_{\beta_\gamma, \gamma, \lambda, \Lambda}$.

We start proving that there is $c > 0$ so that for all $x \neq 0$ in Λ

$$\langle \sigma(0)\, \sigma(x) \rangle \leqslant c\gamma^2 \log \gamma^{-1} \tag{2.23}$$

Proceeding as in the proof of Lemma 2.6, by (2.7) there is $c > 0$ so that

$$\langle \sigma(0)\, \sigma(x) \rangle \leqslant \sum_{n \geqslant 1} \beta_\gamma^n J_{\gamma, \Lambda}^n(x, 0) + c\, \frac{\gamma^{100}}{1 - \beta_\gamma} \tag{2.24}$$

As shown in Appendix A, $J_{\gamma, \Lambda}^n(x, 0) \leqslant c\gamma^2 n^{-1}$, (2.23) then follows directly from (2.24).

To prove (2.22) we go back to (2.19). we bound the remainder term (with $h_\gamma(x)^{10}$) as in the proof of Lemma 2.9, for the other correlations appearing in the second to fourth row we use (2.23) and the Gaussian inequalities. Recalling that $J_\gamma(0, x) \leqslant c\gamma^2$ and that the sum over x is equal to 1 we conclude that there is $c > 0$ so that

$$\langle \sigma(0)\, \sigma(x) \rangle \geqslant \beta_\gamma J_\gamma(x, 0) + \beta_\gamma \sum_{y \neq 0} J_\gamma(x, y) \langle \sigma(0)\, \sigma(y) \rangle - c[\gamma^2 \log \gamma^{-1}]^2 \tag{2.25}$$

Let

$$N_\gamma := [\gamma^{-2}(\log \gamma^{-1})^{-200}], \qquad [a] \equiv \text{the integer part of } a$$

and p a positive integer. Then

$$\langle \sigma(0)\, \sigma(x) \rangle \geqslant \sum_{n = p}^{N_\gamma} \beta_\gamma^n J_\gamma^n(x, 0) - N_\gamma c[\gamma^2 \log \gamma^{-1}]^2 \tag{2.26}$$

In Theorem A.1 of the Appendix A it is shown that for a suitable constant c, if $\gamma|x| \leqslant 100$,

$$J_\gamma^n(x, 0) \geqslant \frac{\gamma^2}{2\pi D_\gamma n} \exp\left\{ -\frac{100^2}{2D_\gamma n} \right\} - \frac{c\gamma^2}{n^2} \tag{2.27}$$

with $D_\gamma \to D$ as $\gamma \to 0^+$. Thus for any $\varepsilon > 0$ there is p so that for $n \geqslant p$ and all $\gamma > 0$ small enough,

$$J''_\gamma(x, 0) \geqslant \frac{(1-\varepsilon)\,\gamma^2}{2\pi D n} - \frac{c\gamma^2}{n^2} \tag{2.28}$$

On the other hand given $\varepsilon > 0$ there is γ_ε so that for all $\gamma \leqslant \gamma_\varepsilon$ and $n \leqslant N_\gamma$

$$\beta''_\gamma \geqslant 1 - \varepsilon$$

In conclusion

$$\langle \sigma(0)\,\sigma(x)\rangle \geqslant \frac{(1-\varepsilon)^2\,\gamma^2}{2\pi D} \sum_{n=p}^{N_\gamma} n^{-1} - c'\gamma^2 - N_\gamma c[\gamma^2 \log \gamma^{-1}]^2 \tag{2.29}$$

the term $c'\gamma^2$ bounding the sum of $c\gamma^2 n^{-2}$. By the arbitrarity of ε we then obtain the bound (2.22). Proposition 2.10 is proved. ∎

Proposition 2.11. For any $b < C$ there are $\gamma_b > 0$ and c so that if $\gamma \leqslant \gamma_b$ and $\beta_{b,\gamma}$ is $(100C, \gamma)$-good relative to $\lambda \in \mathscr{L}$ and to a torus Λ of side $\geqslant \gamma^{-2}$, then for any $X \in \Lambda^4$

$$|\langle \sigma(X)\rangle^T_{\beta_{b,\gamma}, \gamma, \lambda, \Lambda}| \leqslant c\gamma^4[\log\log \gamma^{-1}]^2 \tag{2.30}$$

Proof. We shorthand $\langle\cdot\rangle$ for $\langle\cdot\rangle_{\beta_{b,\gamma}, \gamma, \lambda, \Lambda}$ and write β for $\beta_{b,\gamma}$. Let $X \in \Lambda^4$ and $x \in X$. Using the DLR equations we have, analogously to (2.19),

$$\begin{aligned}
\langle \sigma(X)\rangle^T = {} & \beta \sum_{y \notin X} J_\gamma(x, y) \langle \sigma(X\backslash x)\,\sigma(y)\rangle^T \\
& - \beta \sum_{x' \in X\backslash x}\; \sum_{x'' \in X\backslash(x \cup x')} J_\gamma(x, x') \langle \sigma(x')\,\sigma(x'')\rangle \langle \sigma(x')\,\sigma(x''')\rangle \\
& - \frac{\beta^3}{3} \langle \sigma(X\backslash x)\, h_\gamma(x)^3\rangle' + \frac{2\beta^5}{15} \langle \sigma(X\backslash x)\, h_\gamma(x)^5\rangle' \\
& \pm c(\langle h_\gamma(x)^{14}\rangle^{1/2} + \gamma^{100})
\end{aligned} \tag{2.31}$$

where $x''' = X\backslash(x \cup x' \cup x'')$, $\langle \sigma(X\backslash x)\, a\rangle'$, $a = h_\gamma(x)^3, h_\gamma(x)^5$, has the same expression as $\langle \sigma(X\backslash x)\,\sigma(x)\rangle^T$ if $\sigma(x)$ is replaced by a (we use t instead of T to avoid confusions with the higher order truncated correlation functions).

By using the Gaussian inequalities, the assumption that β is $(100C, \gamma)$-good and the properties of J_γ, by an argument similar to that used in the proof of Lemma 2.9 we conclude that there is $c > 0$ so that

$$\langle h_\gamma(x)^{14}\rangle \leqslant c(\gamma^2 \log \gamma^{-1})^7$$
$$\langle \sigma(X\backslash x)\, h_\gamma(x)^3\rangle' \leqslant c(\gamma^2 \log \gamma^{-1})^3$$
$$\langle \sigma(X\backslash x)\, h_\gamma(x)^5\rangle' \leqslant c(\gamma^2 \log \gamma^{-1})^4$$
$$J_\gamma(x, x')\langle \sigma(x')\,\sigma(x'')\rangle\langle \sigma(x')\,\sigma(x''')\rangle \leqslant c\gamma^2(\gamma^2 \log \gamma^{-1})^2$$

Thus for a new suitable constant c

$$|\langle \sigma(X)\rangle^T| \leqslant \beta \sum_{y \notin X} J_\gamma(x, y)\langle \sigma(X\backslash x)\,\sigma(y)\rangle^T + c\gamma^6(\log \gamma^{-1})^2 \quad (2.32)$$

Let $N_\gamma := [\gamma^{-2}(\log \gamma^{-1})^{-100}]$, call $X = \{x_1,..., x_4\}$, $Y = \{y_1,..., y_4\}$,

$$J^n_\gamma(X, Y) = J^n_\gamma(x_1, y_1) \cdots J^n_\gamma(x_4, y_4)$$

then iterating (2.32) $4N_\gamma$ times we get

$$|\langle \sigma(X)\rangle^T| \leqslant \beta^{4N_\gamma} \sum_{Y \in \Lambda^4} J^{N_\gamma}_\gamma(X, Y)\,|\langle \sigma(Y)\rangle^T| + 4N_\gamma c\gamma^6(\log \gamma^{-1})^2 \quad (2.33)$$

By Theorem A.1 in Appendix A, there is $c > 0$ so that

$$J^{N_\gamma}_\gamma(x, y) \leqslant \frac{\gamma^2}{2\pi D_\gamma N_\gamma} \exp\left\{\frac{-(x\gamma)^2}{2D_\gamma N_\gamma}\right\} + \frac{c\gamma^2}{N^2_\gamma[1 + (\gamma x)/\sqrt{N_\gamma}]^4} \quad (2.34)$$

where D_γ, defined in (A.3), converges to D as $\gamma \to 0^+$.

On the other hand denoting by $\{Y_1, Y_2\}$ a partition of Y into 2 atoms of 2 elements each and writing $Y_1 = (y_{1,1}, y_{1,2})$, $Y_2 = (y_{2,1}, y_{2,2})$

$$\begin{aligned} |\langle \sigma(Y)\rangle^T| &\leqslant \sum_{\{Y_1, Y_2\}} \langle \sigma(Y_1)\rangle\langle \sigma(Y_2)\rangle \\ &\leqslant (100C)^2 \sum_{\{Y_1, Y_2\}} \gamma^2\,\delta_\gamma(y_{1,1} - y_{1,2})\,\gamma^2\,\delta_\gamma(y_{2,1} - y_{2,2}) \end{aligned} \quad (2.35)$$

We fix out attention on a single term in (2.35) say $\gamma^2\,\delta_\gamma(y_1 - y_2)$ $\gamma^2\,\delta_\gamma(y_3 - y_4)$ and consider the sum over y_4. We have

$$\begin{aligned} &\sum_{y_4} J^{N_\gamma}_\gamma(x_4, y_4)\,\gamma^2\,\delta_\gamma(y_3 - y_4) \\ &\quad\leqslant \frac{\gamma^2}{2\pi D_\gamma N_\gamma} \sum_{y_4} \exp\left\{\frac{-[(y_4 - x_4)\,\gamma]^2}{2D_\gamma N_\gamma}\right\} \gamma^2\,\delta_\gamma(y_3 - y_4) \\ &\qquad + cN_\gamma^{-1/2}\gamma^2 \log \gamma^{-1}\left(\frac{1}{N_\gamma} \sum_{y_4} \frac{\gamma^2}{1 + [\gamma(y_4 - x_4))/\sqrt{N_\gamma}]^4}\right) \end{aligned} \quad (2.36)$$

Since the bracket term in (2.36) is a bounded function of γ, the last term in (2.36) is bounded by

$$c'\gamma^3(\log \gamma^{-1})^{51} \tag{2.37}$$

We distinguish in the first sum in (2.36) the terms where $|y_4 - y_3| \leqslant \gamma^{-2}(\log \gamma^{-1})^{-100}$. This sum is bounded by

$$c[\gamma^{-2}(\log \gamma^{-1})^{-100}]^2 \frac{\gamma^2}{2\pi D_\gamma N_\gamma} \gamma^2 \log \gamma^{-1} \leqslant c'(\log \gamma^{-1})^{-99} \gamma^2 \tag{2.38}$$

When $|y_4 - y_3| > \gamma^{-2}(\log \gamma^{-1})^{-100}$

$$\delta_\gamma(y_4 - y_3) \leqslant \frac{1}{2} \log \frac{\gamma^{-2}}{1 + [\gamma\{\gamma^{-2}(\log \gamma^{-1})^{-100}\}]^2} \leqslant c \log \log \gamma^{-1} \tag{2.39}$$

On the other hand

$$\sum_{y_4} \frac{\gamma^2}{2\pi D_\gamma N_\gamma} \exp \left\{\frac{-[(y_4 - x_4)\gamma]^2}{2D_\gamma N_\gamma}\right\} \leqslant c \tag{2.40}$$

so that we deduce that there is $c > 0$ so that

$$\sum_{y_4} j_\gamma(x_4, y_4)\, \gamma^2\, \delta_\gamma(y_3 - y_4) \leqslant c\gamma^2 \log \log \gamma^{-1} \tag{2.41}$$

The same bound is obtained when summing over y_2. The sum over y_1 and y_3 is equal to 1 due to the presence of $J_\gamma(x_1, y_1)$ and $J_\gamma(x_3, y_3)$. (2.30) then follows from (2.40) and (2.33). Proposition 2.11 is proved. ▮

Proof of Proposition 2.2. As usual we denote by $\langle \cdot \rangle$ the expectation given $\beta \equiv \beta_{b,\gamma}$, γ, λ and Λ as in the text of the Proposition. We start from (2.18). Let $2c := (C - b)$. By Proposition 2.10, with $\varepsilon = c$, there is $\gamma^* > 0$ so that for all $\gamma \leqslant \gamma^*$

$$\beta - v_\gamma \leqslant 1 - c\gamma^2 \log \gamma^{-1}$$

Then $(\beta - v_\gamma)^n$ is infinitesimal for $n \geqslant \gamma^{-2}(\log \gamma^{-1})^{-\alpha}$, $\alpha < 1$. On the other hand by Proposition 2.11 $\gamma^{-2}\, |\langle \sigma(0)\, \sigma(Y)\rangle^T| \leqslant c\gamma^2 \log \log \gamma^{-1}$, so that $n\gamma^{-2}\, |\langle \sigma(0)\, \sigma(Y)\rangle^T|$ is infinitesimal if $n \leqslant \gamma^{-2}(\log \gamma^{-1})^{-\alpha}$, for any $\alpha > 0$. We thus define

$$N_\gamma := [\gamma^{-2}(\log \gamma^{-1})^{-1/2}]$$

and iterate (2.18) N_γ times. We have

$$\langle \sigma(0)\,\sigma(x)\rangle = \sum_{n=1}^{N_\gamma} (\beta - v_\gamma)^{n-1} (\beta - q_\gamma)\, J_\gamma^n(x, 0) + (\beta - v_\gamma)^{N_\gamma}\, 100C\gamma^2 \log \gamma^{-1}$$
$$+ cN_\gamma(\gamma^4 \log \gamma^{-1} + \gamma^4 + [\gamma \log \gamma^{-1}]^5) \tag{2.42}$$

having bounded the remaining two point correlation function by $100C\gamma^2 \log \gamma^{-1}$ as, by assumption, $\beta = \beta_{h,\gamma}$ is $(100C, \gamma)$-good. There is $c > 0$ so that

$$(\beta - v_\gamma)^{N_\gamma}\, 100C \log \gamma^{-1} \leqslant c \tag{2.43}$$

hence there is a new constant c such that

$$\langle \sigma(0)\,\sigma(x)\rangle \leqslant \sum_{n=1}^{N_\gamma} (\beta - v_\gamma)^{n-1} (\beta - q_\gamma)\, J_\gamma^n(x, 0) + c\gamma^2 \tag{2.44}$$

By Theorem A.1 in Appendix A

$$\sum_{n=1}^{N_\gamma} (\beta - v_\gamma)^{n-1} (\beta - q_\gamma)\, J_\gamma^n(x, 0)$$
$$\leqslant \gamma^2 \sum_{n=1}^{N_\gamma} (\beta - v_\gamma)^{n-1} \left(\frac{1}{2\pi D_\gamma n} \exp\left\{ \frac{-(x\gamma)^2}{2D_\gamma n} \right\} + \frac{c}{n^2[1 + (\gamma x)/\sqrt{n}]^4} \right) \tag{2.45}$$

Since for all $\gamma > 0$ small enough $(\beta - v_\gamma) \leqslant 1$, the left hand side of (2.45) is bounded by

$$\sum_{n=1}^{N_\gamma} \left[\frac{\gamma^2}{2\pi D_\gamma n} \exp\left\{ \frac{-(x\gamma)^2}{2D_\gamma n} \right\} + \frac{c\gamma^2}{n^2[1 + (\gamma x)/\sqrt{n}]^4} \right] =: a_\gamma(r) + b_\gamma(r) \tag{2.46}$$

where $r := \gamma x$. Given γ and r we call $n_\gamma(r)$ the largest integer n such that

$$n \log n \leqslant \frac{r^2}{2D_\gamma}$$

so that the exponential in (2.46) is bounded by n^{-1} for $n \leqslant n_\gamma(r)$. We then have for any r

$$a_\gamma(r) \leqslant \sum_{n=1}^{n_\gamma(r)} \frac{\gamma^2}{2\pi D_\gamma n^2} + \sum_{n=n_\gamma(r)}^{N_\gamma} \frac{\gamma^2}{2\pi D_\gamma n}$$

(the second sum being absent if $n_\gamma(r) > N_\gamma$) and

$$b_\gamma(r) \leqslant c\gamma^2 \sum_{n \geqslant 1} n^{-2} = c'\gamma^2$$

Thus, going back to (2.14), for any $C' > C = 1/(\pi D)$, there is $\gamma^* > 0$ so that for all $\gamma \leqslant \gamma^*$

$$\langle \sigma(0)\, \sigma(x) \rangle \leqslant a_\gamma(\gamma x) + b_\gamma(\gamma x) + c\gamma^2 \leqslant \frac{C'\gamma^2}{2} \log \left\{ \frac{\gamma^{-2}}{1 + r^2} \right\} = C'\delta_\gamma(x)$$

Proposition 2.2 is proved. ∎

3. BOUNDS ON THE ODD CORRELATIONS

In this section we consider the plus Gibbs state on $\mathbb{Z}^2$ with interaction J_γ (i.e. $\lambda \equiv 0$) at the inverse temperature $\beta_{b,\gamma}$ with $b < C$. We simply denote by $\langle \cdot \rangle$ its expectation, write β for $\beta_{b,\gamma}$ and $m := \langle \sigma(x) \rangle$, $x \in \mathbb{Z}^2$. By ferromagnetic inequalities the magnetization m is always non negative and if $m = 0$ then there is only one Gibbs state, hence Theorem 1.1 will be proved once we show that there is $\gamma_b > 0$ such that $m = 0$ if $\gamma \leqslant \gamma_b$.

The heuristic argument in the introduction suggests to consider the DLR identity

$$m = \langle \tanh\{\beta h_\gamma(0)\} \rangle \tag{3.1}$$

($h_\gamma(0)$ is defined as in (2.9) with Λ replaced by $\mathbb{Z}^2$) and to expand the right hand side of (3.1) in powers of $\beta h_\gamma(0)$. An analogous procedure has been already used several times so far and it has always been possible to truncate the expansion with a negligible error. Using the Cauchy Schwartz inequality we had bounds like $c\gamma^n$, with n sufficiently large to make the error negligible, see for instance the proof of Lemma 2.10. Here instead a bound $c\gamma^n$, no matter how large is n, would not be sufficient, as the error must be compared to m (that appears on the left hand side of (3.1)) and m may be arbitrarily small: we will eventually prove that $m = 0$! We can thus only accept bounds proportional to m and a term is negligible if it is proportional to m by a coefficient $c\gamma^n$ with n suitably large. Therefore we cannot afford to truncate the expansion and we will have to consider the full series. As the power series of $\tanh(\cdot)$ has a finite radius of convergence, the DLR identity (3.1) is not the best suited for our strategy and our first step will be to establish another elementary identity which replaces (3.1).

Lemma 3.1. Let $X \in \mathbb{Z}^2$ be a finite set containing 0, then

$$\langle \sigma(X) \rangle = \langle \sigma(X \backslash 0) \sinh \beta h_\gamma(0) \rangle + \langle \sigma(X)[1 - \cosh \beta h_\gamma(0)] \rangle \quad (3.2)$$

Proof. Denote by $\mathbb{E}_{\sigma_{\mathbb{Z}^2 \backslash 0}}$ the conditional expectation given the configuration $\sigma_{\mathbb{Z}^2 \backslash 0}$ on $\mathbb{Z}^2 \backslash 0$. Then

$$\mathbb{E}_{\sigma_{\mathbb{Z}^2 \backslash 0}}(\sigma(0)\, e^{-\beta h_\gamma(0)\, \sigma(0)}) = \frac{1}{Z(\sigma_{\mathbb{Z}^2 \backslash 0})} \sum_{\sigma(0) = \pm 1} \sigma(0) = 0$$

with $Z(\sigma_{\mathbb{Z}^2 \backslash 0})$ the partition function. Thus, recalling that $0 \in X$,

$$\mathbb{E}(\sigma(X)\, e^{-\beta h_\gamma(0)\, \sigma(0)}) = \mathbb{E}(\sigma(X \backslash 0)\, \sigma(0)\, e^{-\beta h_\gamma(0)\, \sigma(0)}) = 0$$

Since $\sigma(0) = \pm 1$

$$\sigma(0)\, e^{-\beta h_\gamma(0)\, \sigma(0)} = \frac{1 + \sigma(0)}{2}\, e^{-\beta h_\gamma(0)} - \frac{1 - \sigma(0)}{2}\, e^{\beta h_\gamma(0)}$$

Using the two previous identities we get (3.2). Lemma 3.1 is proved. ∎

Using the Notation 2.7 and recalling that q_γ is defined in (2.14), we set

$$p_\gamma := -\frac{\beta^2}{2} \left(\sum_{Y \in S^2} J_\gamma(0, Y) \langle \sigma(Y) \rangle + 2 \sum_{x \neq 0} J_\gamma(0, x) \langle \sigma(0)\, \sigma(x) \rangle \right) + \frac{q_\gamma}{2} \quad (3.3)$$

We will see afterwards that p_γ is negative and this will play a crucial role as it will cause an effective reduction of the inverse temperature, cf. the first term on the right hand side of (3.5) below. We also define for $Y \in S^3$

$$\langle \sigma(Y) \rangle^T := \langle \sigma(Y) \rangle - \sum_{\{Y_1, Y_2\}} \langle \sigma(Y_1) \rangle \langle \sigma(Y_2) \rangle \quad (3.4)$$

where the sum is over the all the partitions of Y into two atoms, one of which being a singleton. Analogously to Lemma 2.9 we have:

Lemma 3.2. There is a constant $c > 0$ such that

$$m = (\beta + p_\gamma)\, m - \frac{\beta^3}{6} \sum_{Y \in S^3} J_\gamma(0, Y) \langle \sigma(Y) \rangle^T$$

$$+ \sum_{k \geq 4} \frac{\beta^k}{k!} \langle (-\sigma(0))^{k+1}\, h_\gamma(0)^k \rangle \pm c\gamma^2 m \quad (3.5)$$

Proof. We set $X=\{0\}$ in (3.2) and expand $\cosh(\beta h_\gamma(0))$ and $\sinh(\beta h_\gamma(0))$ in power series:

$$m=\beta\langle h_\gamma(0)\rangle-\frac{\beta^2}{2!}\langle\sigma(0)\,h_\gamma(0)^2\rangle+\frac{\beta^3}{3!}\langle h_\gamma(0)^3\rangle$$
$$+\sum_{k=4}^{\infty}\frac{\beta^k}{k!}\langle(-\sigma(0))^{k+1}\,h_\gamma(0)^k\rangle \tag{3.6}$$

Recalling that $\sum J_\gamma(x,y)=1$ and using that the plus Gibbs state is translationally invariant we have $m=\langle h_\gamma(0)\rangle$. Since $J_\gamma(0,y)^2\leqslant c\gamma^2 J_\gamma(0,y)$,

$$\langle\sigma(0)\,h_\gamma(0)^2\rangle=\sum_{Y\in(S\backslash 0)^2}J_\gamma(0,Y)\langle\sigma(0)\,\sigma(Y)\rangle\pm cm\gamma^2 \tag{3.7}$$

Setting $Y:=\{y_1,y_2\}$ we have

$$\langle\sigma(0)\,\sigma(Y)\rangle=\langle\sigma(0)\,\sigma(Y)\rangle^T+m\{\langle\sigma(Y)\rangle$$
$$+\langle\sigma(0)\,\sigma(y_1)\rangle+\langle\sigma(0)\,\sigma(y_2)\rangle\}$$

Calling $\{\cdot\}$ the term in curly brackets,

$$\sum_{Y\in(S\backslash 0)^2}J_\gamma(0,Y)\{\cdot\}=\sum_{Y\in(S\backslash 0)^2}J_\gamma(0,Y)\langle\sigma(Y)\rangle$$
$$+2\sum_{y\neq 0}J_\gamma(0,y)\langle\sigma(0)\,\sigma(y)\rangle\pm c\gamma^2$$

with $c>0$ a suitable constant.

The right hand side reconstructs the bracket term on the right hand side of (3.3) with an error $c\gamma^2$ compatible with (3.5). An analogous procedure applied to $\langle h_\gamma(0)^3\rangle$ gives p_γ, thus proving (3.5). Lemma 3.2 is proved. ▮

We will prove that both the series in (3.5) and the term with the truncated correlation function are small in comparison to the other terms. The proof uses again the Gaussian inequalities proved by Newman:

Theorem 3.3. Let $Y\in S^{2n+1}$, $n\geqslant 1$, then (recall that the expectation refers to the plus Gibbs state)

$$\langle\sigma(Y)\rangle\leqslant\sum_{\{Y_1,\ldots,Y_{n+1}\}}\langle\sigma(Y_1)\rangle\cdots\langle\sigma(Y_{n+1})\rangle \tag{3.8}$$

where the sum is over all the partitions of the set Y into atoms $Y_1,\ldots,Y_{n+1}$ of two elements each except one consisting of a singleton.

Remarks. As each of the terms in the sum in (3.8) has a factor m, $\langle\sigma(Y)\rangle$ is bounded proportionally to m with a coefficient that is a product of two point correlations. By the positivity of the correlation functions, the bound (3.8) is a fortiori true if the sites in $Y=\{y_1, y_{2n+1}\}$ are not all distinct.

Using Lemma 3.3 we have:

Lemma 3.4. There is a constant $c>0$ such that

$$\sum_{k\geqslant 4}\frac{\beta^k}{k!}\langle(-\sigma(0))^{k+1}h_\gamma(0)^k\rangle\leqslant cm(\gamma^2\log\gamma^{-1})^2 \tag{3.9}$$

Proof. The terms with k even are non positive, we thus need to consider only the terms $\langle h_\gamma(0)^{2n+1}\rangle$. Let $\{I_1,..., I_{n+1}\}$ be a partition of $\{1,..., 2n+1\}$ into atoms of two elements each, except one with a singleton. Then

$$\langle h_\gamma(0)^{2n+1}\rangle$$
$$\leqslant\sum_{\{I_1,...,I_{n+1}\}}\sum_{X_{I_1},...,X_{I_{n+1}}}J_\gamma(0,X_{I_1})\cdots J_\gamma(0,X_{I_{n+1}})\langle\sigma(X_{I_1})\rangle\cdots\langle\sigma(X_{I_{n+1}})\rangle$$

where $X_{I_1}=\{x_i,x_j\}$ if $I_1=\{i,j\}$ and x_i and x_j vary independently on the whole $\mathbb{Z}^2$ one of the I_j however is a singleton. Consider the case where $I_1=\{i,j\}$. Then by Corollary 2.3 if $C'>C$ there is $\gamma'>0$ and for $\gamma\leqslant\gamma'$

$$\sum_{x_j\in\mathbb{Z}^2}J_\gamma(0,x_j)\langle\sigma(x_i)\,\sigma(x_j)\rangle\leqslant\sum_{x_j\neq x_i}J_\gamma(0,x_j)\,C'\gamma^2\log\gamma^{-1}+c\gamma^2$$

where, by (2.10), $c\gamma^2$ bounds the term with $x_j=x_i$. The right hand side is then bounded by $c'\gamma^2\log\gamma^{-1}$.

Recalling that one of the elements of the partition has a singleton we then get

$$\sum_{X_{I_1},...,X_{I_{n+1}}}J_\gamma(0,,X_{I_1})\cdots J_\gamma(0,X_{I_{n+1}})\langle\sigma(X_{I_1})\rangle\cdots\langle\sigma(X_{I_{n+1}})\rangle$$
$$\leqslant m(c'\gamma^2\log\gamma^{-1})^n$$

The number of partitions $\{I_1,..., I_{n+1}\}$ is $(2n+1)!!$. Lemma 3.4 is proved. ▮

We will next bound the term with the truncated correlation function:

Lemma 3.5. There is $c>0$ such that for all $X\in S^3$

$$|\langle\sigma(X)\rangle^T|\leqslant cm\gamma^2\log\log\gamma^{-1} \tag{3.11}$$

Proof. let $X=(x, y, z) \in S^3$, a a function of the configuration σ and let $\langle \sigma(x)\, \sigma(y)\, a\rangle'$ have the same expression as $\langle \sigma(x)\, \sigma(y)\, \sigma(z)\rangle'$ when $\sigma(x) \to a$.

Then, analogously to (2.31) but starting from (3.2),

$$\langle \sigma(x)\, \sigma(y)\, \sigma(z)\rangle^T = \beta \left\{ \sum_{z' \neq x, y} J_\gamma(z, z') \langle \sigma(x)\, \sigma(y)\, \sigma(z')\rangle^T - m\langle \sigma(x)\, \sigma(y)\rangle [J_\gamma(z, x) + J_\gamma(z, y)] \right\} + \sum_{k=2}^{\infty} \frac{\beta^k}{k!} \langle \sigma(x)\, \sigma(y) \{(-\sigma(z))^{k+1} h_\gamma(z)^k\}\rangle' \tag{3.12}$$

Similarly to (3.9), we get

$$\langle \sigma(x)\, \sigma(y)\, \sigma(z)\rangle^T = \beta \sum_{z' \neq x, y} J_\gamma(z, z') \langle \sigma(x)\, \sigma(y)\, \sigma(z')\rangle^T \pm cm \left(\gamma^4 \log \gamma^{-1} + \left(\gamma^2 \log \frac{1}{\gamma} \right)^2 \right) \tag{3.13}$$

Let

$$N_\gamma := \gamma^{-2} (\log \gamma^{-1})^{-100} \tag{3.14}$$

rename $X=(x_1, x_2, x_3)$, call $Y=(y_1, y_2, y_3) \in S^3$, then iterating $3N_\gamma$ times (3.13) we get

$$\langle \sigma(X)\rangle^T = \beta^{3N_\gamma} \sum_{Y \in S^3} J_\gamma^{N_\gamma}(X, Y) \langle \sigma(Y)\rangle^T \pm cm\gamma^2 (\log \gamma^{-1})^{-98} \tag{3.15}$$

The argument is now similar to that in the proof of Proposition 2.11 after (2.33). Like in (2.35) we have

$$|\langle \sigma(Y)\rangle^T| \leqslant m \sum_{i=1}^{3} \langle \sigma(Y \backslash y_i)\rangle$$

Take now for instance the term with $i=1$ which (3.15) is multiplied by $J_\gamma^{N_\gamma}(x_3, y_3)$. We thus have

$$m \sum_{y_3 \neq y_1, y_2} J_\gamma^{N_\gamma}(x_3, y_3) \langle \sigma(y_3)\, \sigma(y_2)\rangle \leqslant m \sum_{y_3 \neq y_2} J_\gamma^{N_\gamma}(x_3, y_3)\, \gamma^2\, \delta_\gamma(y_3 - y_2)$$

Then by (2.41)

$$|\langle \sigma(X) \rangle^T| \leqslant c'm\gamma^2 \log\log \gamma^{-1} + cm\gamma^2(\log \gamma^{-1})^{-98}$$

Lemma 3.5 is proved. ∎

Proof of Theorem 1.1. Let $b < C$ as in the text of Theorem 1.1 and recall that we are shorthanding $\beta = \beta_{b,\gamma}$. Going back to (3.5) and using Lemma 3.4 and 3.5 there is $c > 0$ and $\gamma^{(1)} > 0$ so that for all $\gamma \leqslant \gamma^{(1)}$

$$m = (\beta + p_\gamma)\, m \pm cm\gamma^2 \log\log \gamma^{-1}$$

By (2.22) (and using the ferromagnetic inequalities to replace the expectation with one referring to the plus state) for any $C' \in (b, C)$ and $C'' > C$ there is $\gamma^{(2)} \in (0, \gamma^{(1)}]$ so that for all $\gamma \leqslant \gamma^{(2)}$ and all $|x| \leqslant 100\gamma^{-1}$

$$C''\gamma^2 \log \gamma^{-1} \geqslant \langle \sigma(0)\, \sigma(x) \rangle \geqslant C'\gamma^2 \log \gamma^{-1}$$

the first inequality comes from Theorem 2.4. Then by (3.3) and (2.14)

$$p_\gamma \leqslant -\frac{3\beta^2}{2} C'\gamma^2 \log \gamma^{-1} + \frac{\beta^3}{2} C''\gamma^2 \log \gamma^{-1} + c\gamma^4 \log \gamma^{-1}$$

the last term takes into account the fact that the sum of $J_\gamma(0, Y)$ in (3.3) is not 1. Choosing C' and C'' sufficiently close to C, $p_\gamma \leqslant -b'\gamma^2 \log \gamma^{-1}$ with $b' > b$, thus recalling that $\beta = 1 + b\gamma^2 \log \gamma^{-1}$

$$0 \leqslant m \leqslant [1 - (b' - b)\, \gamma^2 \log \gamma^{-1}]\, m + c'm\gamma^2 \log\log \gamma^{-1}$$

which for γ small enough implies $m = 0$. Theorem 1.1 is proved. ∎

APPENDIX

In this Appendix we prove a local central limit theorem for the iterates of $J_\gamma(x, y)$. The proofs are classical, we have only checked the dependence on γ of the parameters obtaining the uniform bounds used in the text. For future applications we work in $\mathbb{Z}^d$ with d arbitrary.

Theorem A.1. For any positive integer m there is $c > 0$ so that for all $\gamma \in (0, 1]$, all $n \geqslant 1$ and all $x \in \mathbb{Z}^d$

$$J_\gamma^n(0, x) = G_\gamma(x, n) \pm c\left(\frac{\gamma}{\sqrt{n}}\right)^d \frac{1}{n} \frac{1}{1 + (\gamma\, |x|\, n^{-1/2})^m} \tag{A.1}$$

where

$$G_\gamma(x, n) := \frac{\gamma^d}{(2\pi D_\gamma n)^{d/2}} \exp\left\{-\frac{(\gamma x)^2}{2D_\gamma n}\right\} \tag{A.2}$$

$$D_\gamma^{(2n)} := \sum_{y \in \mathbb{Z}^d} (\gamma y)^{(2n)} J_\gamma(0, y), \qquad D_\gamma \equiv D_\gamma^{(2)} \tag{A.3}$$

The remaining of the appendix is devoted to the proof of Theorem A.1 which follows the classical proof of the local central limit theorem for the sum of i.i.d. variables. In our case however the distribution depends on the parameter γ and we want estimates uniform in γ as $\gamma \to 0$. Analogous problem's have been considered in refs. 5 and 1.

Instead of (A.1) we will prove the equivalent bound with the last term replaced by

$$\frac{c\gamma^d}{n^{1+d/2}} \prod_{i=1}^{d} (\max\{1, [\gamma\, |x_i|\, n^{-1/2}]^{2m}\})^{-1} \tag{A.4}$$

Calling $\mathscr{T} \subset \mathbb{R}^d$ the torus $|-\pi, \pi]^d$ we have for any positive integer n

$$J_\gamma^n(0, x) = \frac{1}{|\mathscr{T}|} \int_{\mathscr{T}} dk\, e^{-ikx} \hat{J}_\gamma^n(k) \tag{A.5}$$

where

$$\hat{J}_\gamma(k) := \sum_{y \in \mathbb{Z}^d} e^{iky} J_\gamma(0, y) \tag{A.6}$$

We call q a multi-index, $q = (q_{i_1}, \ldots, q_{i_l})$, $(i_1, \ldots, i_l) \subset \{1, \ldots, d\}$. Given x, $\gamma > 0$ and n we denote by $(i_1, \ldots, i_l)$ the coordinate labels such that $|x_{i_j}| \geq \gamma^{-1}\sqrt{n}$ and set $q^* \equiv (q^*_{i_1}, \ldots, q^*_{i_l})$, $q^*_{i_j} = 2m$. After integrating by parts (A.5) we get

$$J_\gamma^n(0, x) = \left(\frac{\sqrt{n}}{i\gamma x}\right)^{q^*} \frac{1}{|\mathscr{T}|} \int_{\mathscr{T}} dk\, e^{-ikx} \left(\frac{\gamma}{\sqrt{n}}\right)^{q^*} \frac{\partial^{q^*}}{\partial k^{q^*}} \hat{J}_\gamma^n(k) \tag{A.7}$$

We split the integral in (A.7) into four regions:

$$|k| \leq \gamma n^{-1/3},\ \gamma n^{-1/3} < |k| \leq a\gamma,\ a\gamma \leq |k| \leq b\gamma,\ b\gamma < |k| \tag{A.8}$$

where $a \ll 1 \ll b$ are independent of γ and will be fixed later, we will thus write

$$J_\gamma^n(0, x) = \pi_{\gamma, n}^{(1)}(x) + \cdots + \pi_{\gamma, n}^{(4)}(x) \tag{A.9}$$

where $\pi^{(i)}_{\gamma,n}(x)$ is defined by the integral (A.7) extended to the ith region in (A.8).

The region $|k| \leqslant \gamma n^{-1/3}$

We will first prove that there are a positive integer p and $c > 0$ so that

$$\left(\frac{\gamma}{\sqrt{n}}\right)^{q^*} \frac{\partial^{q^*}}{\partial k^{q^*}} \hat{J}_\gamma(k)^n = \left(\frac{\gamma}{\sqrt{n}}\right)^{q^*} \frac{\partial^{q^*}}{\partial k^{q^*}} e^{-n\gamma^{-2}|k|^2 D_\gamma/2} \pm c\,\frac{(\gamma^{-1}|k|\sqrt{n})^p}{n} e^{-n\gamma^{-2}|k|^2 D_\gamma/2} \tag{A.10}$$

It is clear from the error term in (A.10) that in this region the relevant variable is $\gamma^{-1}\sqrt{n}\,k$.

To prove (A.10) we start by a Taylor–Lagrange expansion to 4th order of $\hat{J}(k)$:

$$\hat{J}_\gamma(k) = 1 - (\gamma^{-1}k)^2 \frac{D_\gamma}{2!} \pm \frac{(\gamma^{-1}k)^4}{4!} D^{(4)}_\gamma \tag{A.11}$$

Then

$$\hat{J}_\gamma(k)^n = \exp\left\{-n\gamma^{-2}k^2 \frac{D_\gamma}{2}\right\}\left(1 \pm c\,\frac{(\gamma^{-1}k\sqrt{n})^4}{n}\right) \tag{A.12}$$

with c a suitable constant. In deriving (A.12) we have used that $n(\gamma^{-1}k)^4 < 1$ which follows from the condition $\gamma^{-1}k \leqslant n^{-1/3}$.

After differentiating q^* times on k we get several terms, each with a factor $\hat{J}_\gamma(k)^{n-p}$, p a positive integer. We then use (A.12) and

$$\hat{J}_\gamma(k)^{-p} = \left(1 \pm \frac{(\gamma^{-1}k\sqrt{n})^2}{n}\frac{D_\gamma}{2!}\right)^{-p} = 1 \pm c\,\frac{(\gamma^{-1}k\sqrt{n})^2}{n} \tag{A.13}$$

where c is a suitable constant (that depends on p hence, ultimately on m). Similarly we have for $1 \leqslant i \leqslant d$

$$\frac{\partial}{\partial k_i}\hat{J}_\gamma(k) = -\frac{1}{d}\gamma^{-2}k_i D_\gamma \pm \frac{1}{3!}\gamma^{-4}|k|^3 D^{(4)}_\gamma \tag{A.14}$$

$$\frac{\partial^2}{\partial k^2}\hat{J}_\gamma(k) = -\gamma^{-2}D_\gamma \pm \frac{1}{2!}\gamma^{-4}|k|^2 D^{(4)}_\gamma \tag{A.15}$$

hence

$$\frac{\gamma}{\sqrt{n}}\frac{\partial}{\partial k}\hat{J}_\gamma(k) = -\frac{1}{d}\frac{(\gamma^{-1}k\sqrt{n})\,D_\gamma}{n} \pm \frac{(\gamma^{-1}\,|k|\,\sqrt{n})^3}{n^2}\frac{D_\gamma^{(4)}}{3!} \tag{A.16}$$

$$\frac{\gamma^2}{n}\frac{\partial^2}{\partial k^2}\hat{J}_\gamma(k) = -\frac{D_\gamma}{n} \pm \frac{(\gamma^{-1}k\sqrt{n})^2}{n^2}\frac{D_\gamma^{(4)}}{2} \tag{A.17}$$

Analogously, if $q \leqslant q^*$ is a multi-index, $|q| > 2$ and q_{ev} is the smallest multi-index $\geqslant q$ with even entries

$$\left|\frac{\gamma^q}{n^{q/2}}\frac{\partial^q}{\partial k^q}\hat{J}_\gamma(k)\right| \leqslant c(1 + [\gamma^{-1}\,|k|\,\sqrt{n}]^p)\frac{1}{n^{q_{ev}/2}} \tag{A.18}$$

with p a suitable integer and c a constant (dependent on m). Observe that $q_{ev}/2 \geqslant 2$.

We have now all the ingredients to prove (A.10). In fact the first term in (A.10) takes into account the contribution of the first terms on the r.h.s. of (A.12), (A.13), (A.16) and (A.17). All the other terms have at least a factor n^{-1}, hence (A.10).

Recalling (A.9), we write the contribution $\pi_{\gamma,n}^{(1)}$ to $J_\gamma^n(0, x)$ due to the integral over $|k| \leqslant \gamma n^{-1/3}$ as

$$\begin{aligned}\pi_{\gamma,n}^{(1)}(x) &:= \left(\frac{\sqrt{n}}{i\gamma x}\right)^{q^*}\frac{1}{|\mathscr{T}|}\int_{|k| \leqslant \gamma n^{-1/3}} dk\, e^{-ikx}\left(\frac{\gamma}{\sqrt{n}}\right)^{q^*}\frac{\partial^{q^*}}{\partial k^{q^*}}\hat{J}_\gamma(k)^n \\ &= \pi_{\gamma,n}^{(1,1)}(x) \pm \pi_{\gamma,n}^{(1,2)}(x)\end{aligned} \tag{A.19}$$

$$\pi_{\gamma,n}^{(1,1)}(x) := \left(\frac{\sqrt{n}}{i\gamma x}\right)^{q^*}\frac{1}{|\mathscr{T}|}\int_{|k| \leqslant \gamma n^{-1/3}} dk\, e^{-ikx}\left(\frac{\gamma}{\sqrt{n}}\right)^{q^*}\frac{\partial^{q^*}}{\partial k^{q^*}} e^{-n\gamma^{-2}k^2D_\gamma/2} \tag{A.20}$$

$$\pi_{\gamma,n}^{(1,2)}(x) := \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*}\frac{1}{|\mathscr{T}|}\int_{|k| \leqslant \gamma n^{-1/3}} dk\, c e^{-n\gamma^{-2}k^2D_\gamma/2}\frac{(\gamma^{-1}k\sqrt{n})^p}{n} \tag{A.21}$$

There is a constant $c^{(1,2)}$ so that

$$\pi_{\gamma,n}^{(1,2)}(x) \leqslant c^{(1,2)}\frac{1}{n}\left(\frac{\gamma}{\sqrt{n}}\right)^d\left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \tag{A.22}$$

which is obtained from (A.21) by the change of variable $k \to \sqrt{n}\,\gamma^{-1}k$.

The integrand in (A.20) after the differentiation is bounded by a polynomial $\Gamma(\sqrt{n}\,\gamma^{-1}k)$. In (A.20) we then extend the integral to the whole torus $\mathscr{T}$ and write

$$\pi_{\gamma,n}^{(1,1)}(x)=\left(\frac{\sqrt{n}}{i\gamma x}\right)^{q^*}\frac{1}{|\mathscr{T}|}\int_{\mathscr{T}}dk\,e^{-ikx}\left(\frac{\gamma}{\sqrt{n}}\right)^{q^*}\frac{\partial^{q^*}}{\partial k^{q^*}}e^{-n\gamma^{-2}k^2D_\gamma/2}$$
$$\pm\left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*}\frac{1}{|\mathscr{T}|}\int_{|k|>\gamma n^{-1/3}}dk\,e^{-n\gamma^{-2}k^2D_\gamma/2}\Gamma(\sqrt{n}\,\gamma^{-1}k)$$

The first term is then integrated by parts giving the main contribution to $J_\gamma^n(0,x)$, i.e. the Gaussian term $G_\gamma(x,n)$. The other term is exponentially small:

$$\pi_{\gamma,n}^{(1,1)}(x)=G_\gamma(x,n)\pm c^{(1,1)}\left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*}e^{-\lambda n^{1/3}}\left(\frac{\gamma}{\sqrt{n}}\right)^d \tag{A.23}$$

$$\lambda:=\frac{1}{4}\inf_{\gamma\in(0,1)}D_\gamma>0 \tag{A.24}$$

$$c^{(1,1)}\geqslant\frac{1}{|\mathscr{T}|}\int_{\mathscr{T}}dk\,e^{-n\gamma^{-2}k^2D_\gamma/4}\Gamma(\sqrt{n}\,\gamma^{-1}k) \tag{A.25}$$

In conclusion

$$\pi_{\gamma,n}^{(1)}(x)=G_{\gamma,n}(x)\pm\left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*}\left(\frac{\gamma}{\sqrt{n}}\right)^d\left(c^{(1,1)}e^{-\lambda n^{1/3}}+\frac{c^{(1,2)}}{n}\right) \tag{A.26}$$

The region $\gamma n^{-1/3}\leqslant|k|\leqslant\gamma a$

Let $a>0$ be small enough, λ as in (A.24). Then from (A.11) for $k\leqslant\gamma a$

$$\hat{J}_\gamma(k)\leqslant 1-(\gamma^{-1}k)^2\lambda \tag{A.27}$$

By (A.16)–(A.18) there is c so that

$$\left|\gamma^q\frac{\partial^q}{\partial k^q}\hat{J}_\gamma(k)\right|\leqslant c \tag{A.28}$$

Recalling (A.9)

$$\pi_{\gamma,n}^{(2)}(x):=\left(\frac{\sqrt{n}}{i\gamma x}\right)^{q*}\frac{1}{|\mathscr{T}|}\int_{\gamma n^{-1/3}\leqslant|k|\leqslant\gamma a}dk\,e^{-ikx}\left(\frac{\gamma}{\sqrt{n}}\right)^{q^*}\frac{\partial^{q^*}}{\partial k^{q^*}}\hat{J}_\gamma(k)^n \tag{A.29}$$

Then there is c so that

$$\pi^{(2)}_{\gamma,n}(x) \leqslant \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \frac{1}{|\mathscr{T}|} \int_{\gamma n^{-1/3} \leqslant |k| \leqslant \gamma a} dk\, n^{|q^*|/2} c[1-(\gamma^{-1}k)^2 \lambda]^{n-|q^*|}$$

Hence there is $c^{(2)}$ so that

$$\pi^{(2)}_{\gamma,n}(x) \leqslant \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \left(\frac{\gamma}{\sqrt{n}}\right)^{d} n^{|q^*|/2} e^{-n^{-1/3}\lambda^2/2} \tag{A.30}$$

The region $\gamma a \leqslant |k| \leqslant \gamma b$

For any $b > a$

$$\lim_{\gamma \to 0^+} \sup_{a \leqslant |\gamma^{-1}k| \leqslant b} |\hat{J}_\gamma(k) - \hat{J}(k)| = 0 \tag{A.31}$$

where $\hat{J}(k)$ is the Fourier transform of $J(r)$. Then there is $\lambda_{a,b} \in (0, 1)$ and $\gamma_{a,b}$ so that for all $\gamma \leqslant \gamma_{a,b}$

$$\hat{J}_\gamma(k) \leqslant \lambda_{a,b} \tag{A.32}$$

Recalling (A.9)

$$\pi^{(3)}_{\gamma,n}(x) := \left(\frac{\sqrt{n}}{i\gamma x}\right)^{q^*} \frac{1}{|\mathscr{T}|} \int_{\gamma a \leqslant |k| \leqslant \gamma b} dk\, e^{-ikx} \left(\frac{\gamma}{\sqrt{n}}\right)^{q^*} \frac{\partial^{q^*}}{\partial k^{q^*}} \hat{J}_\gamma(k)^n \tag{A.33}$$

Then by (A.28), analogously to (A.30), we have

$$\begin{aligned} \pi^{(3)}_{\gamma,n}(x) &:= \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \frac{1}{|\mathscr{T}|} \int_{\gamma a \leqslant |k| \leqslant \gamma b} dk\, c n^{|q^*|/2} \lambda_{a,b}^{n-|q^*|} \\ &\leqslant c^{(3)} \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \left(\frac{\gamma}{\sqrt{n}}\right)^{d} n^{(d+|q^*|)/2} e^{-n\lambda_{a,k}} \end{aligned} \tag{A.34}$$

The region $|k| \geqslant \gamma b$

By Lemma 5.1 in ref. 5 for any multi-index q

$$\hat{J}_\gamma(k) = \left(\frac{\gamma}{ik}\right)^{q} \left(\frac{ik}{1-e^{ik}}\right)^{q} \sum_y e^{-iky}\, \partial^q_y J_\gamma(0, y) \tag{A.35}$$

where ∂_y^q is the qth-discrete derivative with respect to y:

$$\frac{\partial}{\partial_y} J_\gamma(0, y) = \gamma^{-1}[J_\gamma(0, y) - J_\gamma(0, y + e_j)] \tag{A.36}$$

e_j being the unit vector along the jth coordinate direction. Let m an even integer, $\alpha > 0$ and

$$\psi_{m,\alpha}(r) := \prod_{i=1}^{d} (1_{|r_i| < \alpha} + 1_{|r_i| \geq \alpha} |r_i|^{-m}) \tag{A.37}$$

Then for any m there is c so that for all $|k| \geq \gamma b$

$$\hat{J}_\gamma(k)| \leq c\psi_{m,b}(\gamma^{-1}k) \tag{A.38}$$

We then have for a suitable constant c

$$\begin{aligned}\pi_{\gamma,n}^{(4)}(x) &\leq \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \frac{1}{|\mathscr{T}|} \int_{|k| \geq \gamma b} dk\, cn^{|q^*|/2} \psi_{m,b}(\gamma^{-1}k)^n \\ &\leq \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \left(\frac{\gamma}{\sqrt{n}}\right)^d \int_{b \leq |u| < \infty} du\, n^{(d+|q^*|)/2} \psi_{m,b}(u)^n\end{aligned} \tag{A.39}$$

hence

$$\pi_{\gamma,n}^{(4)}(x) \leq c^{(4)} \left(\frac{\sqrt{n}}{\gamma x}\right)^{q^*} \left(\frac{\gamma}{\sqrt{n}}\right)^d n^{(d+|q^*|)/2} e^{-n \log b} \tag{A.40}$$

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